# Problem 4.74

Neither Example 4.4 nor Problem 4.73 actually solved the Schrödinger equation for the Stern–Gerlach experiment. In this problem we will see how to set up that calculation. The Hamiltonian for a neutral, spin-1/2 particle traveling through a Stern–Gerlach device is

$$H = \frac{p^2}{2m} - \gamma\, \mathbf{B} \cdot \mathbf{S}$$

where **B** is given by Equation 4.169. The most general wave function for a spin-1/2 particle—including both spatial and spin degrees of freedom—is[76]

$$\boldsymbol{\Psi}(\mathbf{r}, t) = \Psi_+(\mathbf{r}, t)\chi_+ + \Psi_-(\mathbf{r}, t)\chi_-,$$

**(a)** Put $\boldsymbol{\Psi}(\mathbf{r}, t)$ into the Schrödinger equation

$$H\boldsymbol{\Psi} = i\hbar \frac{\partial}{\partial t}\boldsymbol{\Psi}$$

to obtain a pair of coupled equations for $\Psi_\pm$. *Partial answer:*

$$-\frac{\hbar^2}{2m}\nabla^2\Psi_+ - \frac{\hbar}{2}\gamma(B_0 + \alpha z)\Psi_+ + \frac{\hbar}{2}\gamma\alpha x\Psi_- = i\hbar\frac{\partial}{\partial t}\Psi_+.$$

**(b)** We know from Example 4.3 that the spin will precess in a uniform field $B_0\hat{k}$. We can factor this behavior out of our solution—with no loss of generality—by writing

$$\Psi_\pm(\mathbf{r}, t) = e^{\pm i\gamma B_0 t/2}\tilde{\Psi}(\mathbf{r}, t).$$

Find the coupled equations for $\tilde{\Psi}_\pm$. *Partial answer:*

$$-\frac{\hbar^2}{2m}\nabla^2\tilde{\Psi}_+ - \frac{\hbar}{2}\gamma\alpha z\tilde{\Psi}_+ + \frac{\hbar}{2}\gamma\alpha x e^{-i\gamma B_0 t}\tilde{\Psi}_- = i\hbar\frac{\partial}{\partial t}\tilde{\Psi}_+.$$

**(c)** If one ignores the oscillatory term in the solution to (b)—on the grounds that it averages to zero (see discussion in Example 4.4)—one obtains uncoupled equations of the form

$$-\frac{\hbar^2}{2m}\nabla^2\tilde{\Psi}_\pm + V_\pm\tilde{\Psi}_\pm = i\hbar\frac{\partial}{\partial t}\tilde{\Psi}_\pm.$$

Based upon the motion you would expect for a particle in the "potential" $V_\pm$, explain the Stern–Gerlach experiment.

[TYPO: Replace $\tilde{\Psi}(\mathbf{r}, t)$ with $\tilde{\Psi}_\pm(\mathbf{r}, t)$.]

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## Solution

[76] In this notation, $|\Psi_+(\mathbf{r})|^2\, d^3\mathbf{r}$ gives the probability of finding the particle in the vicinity of $\mathbf{r}$ with spin up, and similarly measuring its spin along the $z$ axis to be up, and similarly for $|\Psi_-(\mathbf{r})|^2\, d^3\mathbf{r}$ with spin down.

## Part (a)

Substitute the most general form for the wave function into the Schrödinger equation.

$$H\mathbf{\Psi} = i\hbar\frac{\partial}{\partial t}\mathbf{\Psi}$$

$$H(\Psi_+\chi_+ + \Psi_-\chi_-) = i\hbar\frac{\partial}{\partial t}(\Psi_+\chi_+ + \Psi_-\chi_-)$$

$$H\left(\Psi_+\begin{bmatrix}1\\0\end{bmatrix} + \Psi_-\begin{bmatrix}0\\1\end{bmatrix}\right) = i\hbar\frac{\partial}{\partial t}\left(\Psi_+\begin{bmatrix}1\\0\end{bmatrix} + \Psi_-\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$H\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix} = i\hbar\frac{\partial}{\partial t}\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix}$$

Expand the Hamiltonian, noting that the magnetic field in Equation 4.169 on page 174 is $\mathbf{B}(x,y,z) = -\alpha x\hat{\mathbf{x}} + (B_0 + \alpha z)\hat{\mathbf{z}}$.

$$\begin{aligned}
\begin{bmatrix}i\hbar\frac{\partial\Psi_+}{\partial t}\\ i\hbar\frac{\partial\Psi_-}{\partial t}\end{bmatrix} &= H\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix}\\
&= \left(\frac{\hat{p}^2}{2m} - \gamma\,\mathbf{B}\cdot\mathbf{S}\right)\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix}\\
&= \left\{\frac{(-i\hbar\nabla)^2}{2m} - \gamma[-\alpha x\mathsf{S}_x + (B_0+\alpha z)\mathsf{S}_z]\right\}\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix}\\
&= \left\{-\frac{\hbar^2}{2m}\nabla^2 + \gamma\alpha x\frac{\hbar}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} - \gamma(B_0+\alpha z)\frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right\}\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix}\\
&= \left\{-\frac{\hbar^2}{2m}\nabla^2 + \frac{\hbar\gamma}{2}\begin{bmatrix}-B_0-\alpha z & \alpha x\\ \alpha x & B_0+\alpha z\end{bmatrix}\right\}\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix}\\
&= -\frac{\hbar^2}{2m}\nabla^2\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix} + \frac{\hbar\gamma}{2}\begin{bmatrix}-B_0-\alpha z & \alpha x\\ \alpha x & B_0+\alpha z\end{bmatrix}\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix}\\
&= -\frac{\hbar^2}{2m}\nabla^2\begin{bmatrix}\Psi_+\\\Psi_-\end{bmatrix} + \frac{\hbar\gamma}{2}\begin{bmatrix}(-B_0-\alpha z)\Psi_+ + \alpha x\Psi_-\\ \alpha x\Psi_+ + (B_0+\alpha z)\Psi_-\end{bmatrix}\\
&= \begin{bmatrix}-\frac{\hbar^2}{2m}\nabla^2\Psi_+ + \frac{\hbar\gamma}{2}[(-B_0-\alpha z)\Psi_+ + \alpha x\Psi_-]\\ -\frac{\hbar^2}{2m}\nabla^2\Psi_- + \frac{\hbar\gamma}{2}[\alpha x\Psi_+ + (B_0+\alpha z)\Psi_-]\end{bmatrix}
\end{aligned}$$

As a result, the Schrödinger equation splits into two equations, both involving $\Psi_+$ and $\Psi_-$.

$$\begin{cases} i\hbar\dfrac{\partial\Psi_+}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\Psi_+ + \dfrac{\hbar\gamma}{2}[(-B_0 - \alpha z)\Psi_+ + \alpha x\Psi_-] \\ i\hbar\dfrac{\partial\Psi_-}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\Psi_- + \dfrac{\hbar\gamma}{2}[\alpha x\Psi_+ + (B_0 + \alpha z)\Psi_-] \end{cases}$$

**Part (b)**

Make the substitutions,

$$\begin{cases} \Psi_+ = e^{i\gamma B_0 t/2}\tilde{\Psi}_+ \\ \Psi_- = e^{-i\gamma B_0 t/2}\tilde{\Psi}_- \end{cases},$$

in the system.

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\left(e^{i\gamma B_0 t/2}\tilde{\Psi}_+\right) = -\dfrac{\hbar^2}{2m}\nabla^2\left(e^{i\gamma B_0 t/2}\tilde{\Psi}_+\right) + \dfrac{\hbar\gamma}{2}\left[(-B_0 - \alpha z)\left(e^{i\gamma B_0 t/2}\tilde{\Psi}_+\right) + \alpha x\left(e^{-i\gamma B_0 t/2}\tilde{\Psi}_-\right)\right] \\ i\hbar\dfrac{\partial}{\partial t}\left(e^{-i\gamma B_0 t/2}\tilde{\Psi}_-\right) = -\dfrac{\hbar^2}{2m}\nabla^2\left(e^{-i\gamma B_0 t/2}\tilde{\Psi}_-\right) + \dfrac{\hbar\gamma}{2}\left[\alpha x\left(e^{i\gamma B_0 t/2}\tilde{\Psi}_+\right) + (B_0 + \alpha z)\left(e^{-i\gamma B_0 t/2}\tilde{\Psi}_-\right)\right] \end{cases}$$

$$\begin{cases} i\hbar\left(e^{i\gamma B_0 t/2}\dfrac{\partial\tilde{\Psi}_+}{\partial t} + \cancel{\dfrac{i\gamma B_0}{2}e^{i\gamma B_0 t/2}\tilde{\Psi}_+}\right) = -\dfrac{\hbar^2}{2m}e^{i\gamma B_0 t/2}\nabla^2\tilde{\Psi}_+ - \cancel{\dfrac{\hbar\gamma B_0}{2}e^{i\gamma B_0 t/2}\tilde{\Psi}_+} - \dfrac{\hbar\gamma\alpha}{2}ze^{i\gamma B_0 t/2}\tilde{\Psi}_+ + \dfrac{\hbar\gamma\alpha}{2}xe^{-i\gamma B_0 t/2}\tilde{\Psi}_- \\ i\hbar\left(e^{-i\gamma B_0 t/2}\dfrac{\partial\tilde{\Psi}_-}{\partial t} - \cancel{\dfrac{i\gamma B_0}{2}e^{-i\gamma B_0 t/2}\tilde{\Psi}_-}\right) = -\dfrac{\hbar^2}{2m}e^{-i\gamma B_0 t/2}\nabla^2\tilde{\Psi}_- + \dfrac{\hbar\gamma\alpha}{2}xe^{i\gamma B_0 t/2}\tilde{\Psi}_+ + \cancel{\dfrac{\hbar\gamma B_0}{2}e^{-i\gamma B_0 t/2}\tilde{\Psi}_-} + \dfrac{\hbar\gamma\alpha}{2}ze^{-i\gamma B_0 t/2}\tilde{\Psi}_- \end{cases}$$

Divide both sides of the first equation by $e^{i\gamma B_0 t/2}$, and multiply both sides of the second equation by $e^{i\gamma B_0 t/2}$.

$$\begin{cases} i\hbar\dfrac{\partial\tilde{\Psi}_+}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\tilde{\Psi}_+ - \dfrac{\hbar\gamma\alpha}{2}z\tilde{\Psi}_+ + \dfrac{\hbar\gamma\alpha}{2}xe^{-i\gamma B_0 t}\tilde{\Psi}_- \\ i\hbar\dfrac{\partial\tilde{\Psi}_-}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\tilde{\Psi}_- + \dfrac{\hbar\gamma\alpha}{2}xe^{i\gamma B_0 t}\tilde{\Psi}_+ + \dfrac{\hbar\gamma\alpha}{2}z\tilde{\Psi}_- \end{cases}$$

**Part (c)**

Ignore the terms with complex exponentials on the grounds that they average to zero in order to decouple the equations.

$$\begin{cases} i\hbar\dfrac{\partial\tilde{\Psi}_+}{\partial t} \approx -\dfrac{\hbar^2}{2m}\nabla^2\tilde{\Psi}_+ - \dfrac{\hbar\gamma\alpha}{2}z\tilde{\Psi}_+ \\ i\hbar\dfrac{\partial\tilde{\Psi}_-}{\partial t} \approx -\dfrac{\hbar^2}{2m}\nabla^2\tilde{\Psi}_- + \dfrac{\hbar\gamma\alpha}{2}z\tilde{\Psi}_- \end{cases}$$

A particle in the state $\tilde{\Psi}_+$ (with spin up) is expected to accelerate in the positive $z$-direction because the potential energy decreases with increasing $z$. Similarly, a particle in the state $\tilde{\Psi}_-$ (with spin down) is expected to accelerate in the negative $z$-direction because the potential energy decreases with decreasing $z$.